1 Exercises in discrete choice, entropy and gravity (2)

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1. Consider the choice between buying or not buying a certain good. This choice can be modelled by a binary logit model:

$$P = \frac{e^{K-\mu c}}{1+e^{K-\mu c}}$$

where P is the probability to buy the good, c is the price of the good and K and μ are constant parameters.

   (a) Draw a graph of P as a function of c. Give some important properties of the graph.

   (b) Show how the appearance of the curve changes as K or μ changes, respectively.

   (c) Assume we have many individuals n, each with their own parameter values μ_n and K_n. We are interested in the aggregated demand

$$P = \frac{1}{N}\sum_n \frac{e^{K_n-\mu_n c}}{1+e^{K_n-\mu_n c}}$$

   Discuss how the appearance of the curve is affected by the variance in μ_n and K_n.

2. In a town there are N libraries. An individual is going to choose which library he/she is going to visit. The utility from visiting library i ($i = 1, 2, ..., N$) is assumed to be

$$U_i = \ln B_i - c_i + \varepsilon_i$$

where B_i is the number of books, c_i is the travel cost and ε_i is a random, Gumbel$(0, 1)$ distributed variable. The individual is assumed to choose the library that maximises his/her utility.

   (a) What is the probability (you do not have to derive it) that the individual chooses library i?

   (b) Assume that the number of books in all libraries is redoubled. What will the probability to choose library i be then?

   (c) How much will the individual's expected utility change in the latter case?

(d) One individual happens to live so that the travel cost is the same to two of the libraries. Now there is a proposal to shut down one of these and move all the books to the other one. What would the probability to visit the one remaining library be, expressed in the probabilities to visit the two old ones? What would the effect on the expected utility of the individual be?

3. You are going to construct a transport demand model for a town. As a part of this you are going to make a travel demand survey and estimate a mode choice model in this form:

$$p_m^n = \frac{e^{\alpha_m + \beta t_m^n + \gamma c_m^n}}{\sum_k e^{\alpha_k + \beta t_k^n + \gamma c_k^n}}$$

where t_m^n is the travel time and c_m^n the travel cost with mode m for individual n in the survey.

(a) What information about the individuals, except travel time and travel cost, do you need to have in order to estimate the actual model?

(b) State the maximum likelihood-problem for estimation of the parameters α_m, β, and γ.

(c) Define the direct elasticity for p_m^n with respect to the travel cost c_m^n, and derive an expression for this elasticity.

4. Show that the solution to the optimisation problem

$$\max_{N_i} - \sum_i N_i \ln N_i$$

such that

$$\begin{aligned} \sum_i N_i t_i &= N\bar{t} \\ \sum_i N_i &= N \end{aligned}$$

where N and $\bar{t}$ are given constants can be written in the form

$$N_i = N \frac{e^{\beta t_i}}{\sum_j e^{\beta t_j}}$$

This could be a model for the distribution of N citizens on residential areas i in a town where all workplaces are situated in the city centre, to which the travel time from residential area i is t_i, and where the observed mean travel time to the centre over all individuals is $\bar{t}$. State with a motivation how the sign of β depends on the values of these variables.

5. Consider an individual who will choose mode of transport for a trip. There are $m = 1, 2, ..., M$ modes to choose among. The probability that the individual will choose mode m is

$$p_m = \frac{\exp(a_m - \beta t_m - \lambda c_m)}{\sum_{n=1}^{N} \exp(a_n - \beta t_n - \lambda c_n)}$$

   (a) Formulate a random utility maximixation problem that will imply these choice probabilities.

   (b) Express in words the meaning of the cross elasticity of a choice probability p_m with respect to the travel time t_n for a different mode n.

   (c) Derive a mathematical expression for this cross elasticity.

6. Assume that an individual has the following probability for the combined choice of a destination d and a mode m:

$$p_{dm} = \frac{\exp(V_{dm})}{\sum_{d'=1}^{D} \sum_{m'=1}^{M} \exp(V_{d'm'})}$$

   (a) Show that this probability can be expressed as the product of a probability of choosing destination d and a probability of choosing mode m conditional on the choice of destination d as follows:

$$p_{dm} = p_d \cdot p_{m|d}$$

   where

$$p_d = \frac{\exp(\widetilde{V_d})}{\sum_{d'} \exp(\widetilde{V_{d'}})}$$

   (b) Specify $\widetilde{V_d}$.

7. The following equation is included in a model for transport and location analysis:

$$S_j = N_S \frac{\exp(V_j)}{\sum_k \exp(V_k)}$$

where

$N_S =$	total number of shops in the region
$V_j =$	$\beta_c \ln T_j + \beta_w U_j + \beta_L \ln(1/S_j)$
$U_j =$	$\ln \sum_i W_i \exp(\sigma(t_{ij} + t_{ji}) + \lambda(c_{ij} + c_{ji}))$
$T_j =$	number of shopping trips to zone j
$W_i =$	number of jobs in zone i
$t_{ij} =$	travel time in minutes by car from zone i to zone j
$c_{ij} =$	monetary travel cost in SEK by car from zone i to zone j

β_c, β_w, β_L, σ and λ are estimated parameters.

(a) Explain in words what the model does.

(b) How can the three terms in V_j be justified?

(c) In what way could it be argued that the model takes agglomeration effects into account?

(d) Explain with some brief justification what sign each of the parameters β_c, β_w, β_L, σ and λ should have.

8. In a town you can either go by car or by bus. A travel demand survey is carried out considering the choice of mode.

   (a) Set up a simple logit model for the mode choice, where you have included some suitable variables and parameters.

   (b) Set up the maximum likelihood-problem for the estimation of one of the included parameters. State also which data that are needed to estimate all the parameters in your model.

   (c) Derive the direct elasticity for the car probability with respect to car travel cost (which should be among the variables you included in 8a).

9. Consider a town with a centre where all jobs are located. Outside the centre there are n equally large residential areas indexed $i = 1, 2, ..., n$. The commuting cost from residential area i to the centre is c_i. In the town there are N households, and in every household there is one worker commuting daily into the centre. The average commuting cost is known to be $\bar{c}$. We now want to set up a model for how many households N_i are living in each residential area i. Derive a gravity model for this problem according to the theory of entropy maximisation. If you need to include some Lagrange parameter that you cannot solve for, specify how it can be determined.

10. Show how to derive the so-called logsum, weighing together the travel costs for different modes, from a doubly constrained gravity model with and without modal split, T_{ij}^k and T_{ij}.

11. Consider a logit model

$$P_i = \frac{\exp(A_i - \alpha t_i + \beta t_i^2)}{\sum_{j=1}^{I} \exp(A_j - \alpha t_j + \beta t_j^2)}, \quad i = 1, ..., I$$

(a) Derive the direct elasticity $E_i = \frac{\partial P_i / \partial t_i}{P_i / t_i}$ for this model.

(b) Derive the cross elasticity $E_{ij} = \frac{\partial P_i / \partial t_j}{P_i / t_j}$, where $j \neq i$, for this model.

(c) Interpret the definitions of the elasticities in words.